

ON THE SLOPE OF BIELLIPTIC FIBRATIONS

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Abstract

Let $\pi : S \rightarrow B$ be a bielliptic fibration. We prove S is, up to base change, a rational double cover of an elliptic fibration and that π is isotrivial provided it is smooth. Finally, we prove that the slope of π is at least four provided the genus of the fibre is at least six

A la memoria de Fernando

0 Introduction

Let $\pi : S \rightarrow B$ be a *fibration*, i.e. a surjective morphism with connected fibres, from a smooth surface S onto a smooth curve B . A fibration is said to be *relatively minimal* when it has no vertical (-1) -curve. Let g denote the genus of a general fibre and b the genus of B .

Let $\omega_{S/B} = \omega_S \otimes \pi^*(\omega_B^{-1})$ be the relative canonical bundle and let $\Delta(\pi) := \deg \pi_*(\omega_{S/B})$. It is known that $\Delta(\pi) \geq 0$ and that $\Delta(\pi) = 0$ if and only if π is locally trivial. Assume π is not locally trivial. Then we define the *slope* of π as

$$\lambda(\pi) := \omega_{S/B}^2 / \Delta(\pi)$$

(see [19]). There are several results on the lower slope of relatively minimal fibrations of genus $g \geq 2$. First of all we have $\lambda \geq 4 - \frac{4}{g}$ (see [8], [12], [13], [18] for the hyperelliptic case and [19] for the general case) and equality holds only in the hyperelliptic case ([9]). There are improvements in the non-hyperelliptic case for $g \leq 5$ (see [4], [7], [9], [11], [14]) but the presently known techniques seem to have some limitations to extend these results to higher genus.

Recently Konno is trying to find good bounds depending on some extra numerical invariants of the general fibre, such as the Clifford index. In [10], Konno finds better bounds for trigonal and plane quintic fibrations (so Clifford index 1), although they do not seem to be sharp. Also in [11] he gets general bounds depending on the Clifford index in some cases.

In this paper we deal with the case of bielliptic fibrations (i.e., when the general fibre has a 2-to-1 map onto an elliptic curve). Using the glueing results of [2] we know that,

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if $g \geq 6$ a bielliptic fibration is a (generically) double cover of an elliptic fibration. We prove (example 1.2) that this is not true in general if $g \leq 5$ due to the existence of several bielliptic maps in the general fibre.

Using this we get the following sharp bound for the slope of bielliptic fibrations.

Theorem 2.1 *Let $\pi : S \longrightarrow B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let V be the relative minimal model of the elliptic fibration obtained in section 1. Then*

- (a) $\lambda(\pi) \geq 4 + \frac{2(g-5)\mathcal{X}\mathcal{O}_V}{\Delta(\pi)} \geq 4.$
- (b) $\lambda(\pi) = 4$ if and only if S is the minimal desingularization of a double cover $S_0 \longrightarrow V$ of a smooth elliptic surface such that
 - All the fibres of the elliptic fibration $\tau : V \longrightarrow B$ are smooth and isomorphic.
 - The branch divisor of the double cover has only negligible singularities.

In particular, the bound is sharp.

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All throughout this paper we work over the field of complex numbers \mathbb{C} .

1 Bielliptic fibrations

Let F be a smooth curve of genus g . F is called *bielliptic* if F admits a 2-to-1 map onto an elliptic smooth curve E . Such a map is always given by the quotient by an involution $i \in \text{Aut}(F)$, called a *bielliptic involution* on F . It is a well known fact that such an involution is unique if $g \geq 6$ (see [1]).

Let $\pi : S \longrightarrow B$ be a fibration of genus g . We say that π is bielliptic if so it is the general fibre F of π . The following result clarifies the structure of such fibrations. Recall that the fibration π is said to be smooth if every fibre is smooth and it is said to be isotrivial if all the smooth fibres are mutually isomorphic.

Proposition 1.1 *Let $\pi : S \longrightarrow B$ be a bielliptic fibration of genus g . Then*

- (a) *Up to base change, S is a rational double cover of an elliptic surface over the base curve.*
- (b) *If $g \geq 6$ the same is true without base change.*
- (c) *If π is smooth, then π is isotrivial.*

Proof. (a) and (b) are consequence of general results given in [2]. We give here a sketch of proof and refer there for details.

Given $\pi : S \rightarrow B$ we can consider $\psi : \underline{\text{Aut}}_{S/B}^{2,2g-2} \rightarrow B$ the scheme of relative automorphisms of S over B of order 2 having $2g - 2$ base points (which corresponds fibrewise to double covers of elliptic curves) which is a quasi-projective B -scheme. After a base change $B' \rightarrow B$ such a map has always a section defined over a non-empty Zariski open subset $U' \subseteq B'$ which corresponds to a rational automorphism Φ of the minimal desingularization S' of $S \times_B B'$ such that $\Phi|_{F_t}$ is a bielliptic involution for $t \in B'$ general. If V is a desingularization of $S'/\langle \Phi \rangle$ we have a rational double cover $S' \dashrightarrow V$ over B' .

If $g \geq 6$ then ψ is clearly 1-to-1 and then base change is not needed in order to have a section.

(c) Isotrivially can be checked after base change. Following [2] section 2 we can consider after base change

$$\begin{array}{ccccc} S & \xrightarrow{i} & J(\pi) & \xrightarrow{f} & J(\pi) \\ & \searrow \pi & \downarrow & \swarrow & \\ & & B & & \end{array}$$

where $J(\pi)$ is the relative Jacobian variety of S over B and f is a rational relative endomorphism of $J(\pi)$ such that $f \circ i$ produces a bielliptic map on the general fibre of π . Let $V = \overline{(f \circ i)(S)}$. Note that V is an elliptic surface over B (possibly singular). Nevertheless classification of singular fibres of a smooth elliptic surface shows, since $J(\pi)_t$ is an abelian variety for every $t \in B$, that V is smooth and the map $\tau : V \rightarrow B$ is also smooth. Moreover, the map $g = f \circ i : S \dashrightarrow V$ can be solved after some blow-ups but then exceptional curves must be contracted since $V \subseteq J(\pi)$. So we have that S is a double cover of a smooth elliptic fibration (perhaps after base change). In particular every fibre of π is bielliptic.

Consider now the double cover $g : S \rightarrow V$. Since g has degree two the branching divisor of g must be smooth and hence it is étale over B . After new base changes we can assume that the irreducible components of the branching divisor D are sections of τ . Moreover, since $\tau : V \rightarrow B$ is a smooth elliptic fibration it is isotrivial and then, after base change, we can assume $V = B \times E$ (E : elliptic smooth curve). Let D_1 be an irreducible component of D . If D_1 is a trivial section of τ then so must be the other components and then π is clearly isotrivial. Assume D_1 is not a trivial section. Then $D_1 = \{(b, \alpha(b)) \in B \times E \mid \alpha : B \rightarrow E \text{ non constant map}\}$. Consider a fixed structure of group on $E = (E, 0)$ and consider the automorphism of V over B : $\beta(b, x) = (b, x + \alpha(b))$. Note that $\beta^{-1}(D_1) = B \times \{0\}$ and, hence, $\beta^{-1}(D)$ is composed of trivial horizontal sections. If we change the base

$$\begin{array}{ccc} S & \xrightarrow{\cong} & S \\ \tilde{g} \downarrow & \otimes & \downarrow g \\ B \times E & \xrightarrow{\beta} & B \times E \end{array}$$

the branching divisor of \tilde{g} is just $\beta^{-1}(D)$ which is trivial. Hence π is isotrivial. \square

A bielliptic curve of genus $g \leq 5$ can have more than one bielliptic involution; the number of such involutions are in correspondence with the elliptic components of $W_4^1(F)$, the Brill-Noether locus of linear series on F of type g_4^1 . We give an example which shows that these involutions do not glue independently for a general fibration.

Example 1.2 Take a genus five curve F with *exactly* two bielliptic involutions $\sigma_i : F \rightarrow E_i$ such that $E_1 \not\cong E_2$, with E_i having no exceptional automorphisms (a count of constants shows that such an F can be chosen). Then we have that $\sigma_1 \times \sigma_2 : F \rightarrow E_1 \times E_2$ embeds F as a smooth curve, $F \in |\ell_1^*(2p_1) \otimes \ell_2^*(2p_2)|$, being $\ell_i : E_1 \times E_2 \rightarrow E_i$ the projections and $(p_1, p_2) \in E_1 \times E_2$. Since $\text{Aut}(E_1 \times E_2)$ acts transitively on $E_1 \times E_2$ we have that for every $(q_1, q_2) \in E_1 \times E_2$ there exists $\tilde{F} \in |\ell_1^*(2q_1) \otimes \ell_2^*(2q_2)|$, $\tilde{F} \cong F$.

Let B be any smooth curve having an involution ι and let $g : B \rightarrow \bar{B} = B_{/\langle \iota \rangle}$. Consider a morphism $\kappa : B \rightarrow \mathbb{P}^1$ with no factorization through \bar{B} . Take a fixed $\bar{t} \in \bar{B}$ such that if $g^{-1}(\bar{t}) = \{t_1, t_2\}$ then $\kappa(t_1) \neq \kappa(t_2)$. After an automorphism of \mathbb{P}^1 we can suppose that $\kappa(t_i)$ is the modular invariant of E_i in $\mathbb{C} \subseteq \mathbb{P}^1$.

Then, by [3] p.160, there exists an elliptic fibration $\tau : V \rightarrow B$ with a section, such that $\tau^{-1}(t_i) \cong E_i$. Let B' be the image in V of the section of τ . Consider the following pull-back

$$\begin{array}{ccc} Z := V \times_B V & \xrightarrow{\xi_2} & V \\ \xi_1 \downarrow & \searrow \xi & \downarrow \tau \\ V & \xrightarrow{\iota \circ \tau} & B \end{array}$$

Then, for $t \in B$ we have $Z_t = \xi^{-1}(t) = E_{\iota(t)} \times E_t$, where $E_m = \tau^{-1}(m)$. The natural involution on $V \times_{\mathbb{C}} V$ induces a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\bar{\iota}} & Z \\ \xi \downarrow & & \downarrow \xi \\ B & \xrightarrow{\iota} & B \end{array}$$

and then

$$\begin{array}{ccc} Z & \xrightarrow{\bar{g}} & \bar{Z} := Z_{/\langle \iota \rangle} \\ \xi \downarrow & & \downarrow \bar{\xi} \\ B & \xrightarrow{g} & \bar{B} \end{array}$$

Note that \bar{Z} is a threefold fibred over \bar{B} and the fibre over $g(t) \in \bar{B}$ general is $E_{\iota(t)} \times E_t$. We can assume \bar{Z} is already smooth.

Let $B'' = \bar{g}(\xi_2^{-1}(B'))$ and $\mathcal{L} = \mathcal{O}_{\bar{Z}}(2B'')$. We have that $\mathcal{L}_{|\bar{Z}_{\bar{t}}} \cong \ell_1^*(2q_1) \otimes \ell_2^*(2q_2)$ for some $(q_1, q_2) \in E_1 \times E_2$. Note that if $\mathfrak{a} \in \text{Pic } \bar{B}$ is ample enough we have an epimorphism

$$H^0(\bar{Z}, \mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})) \rightarrow H^0(E_1 \times E_2, \mathcal{L}_{|\bar{Z}_{\bar{t}}}).$$

Since by hypothesis there exists $F \in |\mathcal{L}_{|\bar{Z}_{\bar{t}}}|$ we get $\bar{S} \in |\mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})|$ a surface fibred over \bar{B} , smooth at a general fibre and such that $\bar{S}_{\bar{t}} = F$. Again, we can suppose \bar{S} is already

smooth. Let $\bar{\pi} : \bar{S} \longrightarrow \bar{B}$ and $F_{\bar{m}} = \bar{\pi}^{-1}(\bar{m})$. For $\bar{m} \in \bar{B}$ general we have that $F_{\bar{m}}$ is a smooth curve of genus 5 having *at least* two bielliptic involutions given by the inclusion $F_{\bar{m}} \subseteq E_{\iota(m)} \times E_m$ (if $g(m) = \bar{m}$) as a $(2, 2)$ -divisor. We claim that for general $\bar{m} \in \bar{B}$, $F_{\bar{m}}$ has exactly two bielliptic involutions. Since this is the case for $F = F_{\bar{t}}$ we only have to prove that having *at most* two of them is an open condition. Consider $W_4^1(\bar{\pi}) \longrightarrow \bar{B}$, the relative Brill-Noether locus of $\bar{\pi}$ (at least over an open set of B , see [16]), after a base change if necessary. The number of bielliptic involutions of $F_{\bar{m}}$ is given by the number of elliptic components of $W_4^1(F_{\bar{m}}) \cong W_4^1(\bar{\pi})_{\bar{m}}$. Then, having at most two of such components is obviously an open condition.

We claim that \bar{S} is not a (birational) double cover of any elliptic fibration $\bar{\tau} : \bar{V} \longrightarrow \bar{B}$. Indeed, assume we have a double cover $\bar{f} : \bar{S} \longrightarrow \bar{V}$ (we can suppose \bar{f} everywhere defined after some blow-ups). Consider the base change diagram

$$\begin{array}{ccc} Z & \longrightarrow & \bar{Z} \\ \uparrow & & \uparrow \\ S & \longrightarrow & \bar{S} \\ \downarrow \tilde{f} & & \downarrow \bar{f} \\ \tilde{V} & \longrightarrow & \bar{V} \\ \downarrow \tilde{\tau} & & \downarrow \bar{\tau} \\ B & \longrightarrow & \bar{B} \end{array}$$

For S we have three double covers of elliptic fibrations over B :

$$\begin{aligned} \tilde{f} : S &\longrightarrow \tilde{V} \\ f_i : S &\longrightarrow V \quad f_i = \xi_i|_S \quad i = 1, 2 \end{aligned}$$

Set $U = \{m \in B \mid E_m \not\cong E_{\iota(m)}; E_m, E_{\iota(m)} \text{ and } \tilde{E}_m \text{ are smooth and } F_m \text{ has exactly two bielliptic involutions}\}$ (where $\tilde{E}_m = \tilde{\tau}^{-1}(m)$). We have that U is a non-empty open set of B . Since $f_1|_{F_m}, f_2|_{F_m}, \tilde{f}|_{F_m}$ are double covers of $E_{\iota(m)}, E_m$ and \tilde{E}_m respectively we have that for every $m \in U$, $\tilde{E}_m \cong E_{\iota(m)}$ or $\tilde{E}_m \cong E_m$.

If $g_1 = g \circ \iota|_U : U \longrightarrow \mathbb{P}^1$, $g_2 = g|_U : U \longrightarrow \mathbb{P}^1$ and $\tilde{g} : U \longrightarrow \mathbb{P}^1$ are the modular morphisms induced by $\iota \circ \tau$, τ and $\tilde{\tau}$ over U respectively we have that $\tilde{g} = g_1$ or $\tilde{g} = g_2$. Assume $\tilde{g} = g_2$.

As we have $t_1, t_2 \in U$ and $\iota(t_1) = t_2$ we get

$$E_{t_1} = \tau^{-1}(t_1) = \tilde{\tau}^{-1}(t_1) \cong \tilde{\tau}^{-1}(t_2) = \tau^{-1}(t_2) = E_{t_2}$$

since $\tilde{\tau}$ is induced by $\bar{\tau} : \bar{V} \longrightarrow \bar{B}$ and then $\tilde{\tau}^{-1}(m) \cong \tilde{\tau}^{-1}(\iota(m))$ for all $m \in B$. But this is imposible since by hypothesis $E_{t_1} = E_1 \not\cong E_2 = E_{t_2}$. \square

2 Double covers and the slope of bielliptic fibrations

We recall some basic facts about double covers (see [6], [3]).

By a double cover we mean a finite, degree two map between surfaces, $f_0 : S_0 \rightarrow V_0$. This map is determined by a divisor Z_0 on V_0 (the branch divisor) and a line bundle \mathcal{L}_0 such that $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(Z_0)$. If V_0 is smooth, S_0 is normal (respectively smooth) if and only if Z_0 is reduced (respectively smooth).

Consider a double cover as above with S_0 normal and V_0 smooth. Then there exists a *canonical resolution of singularities* for S_0 which consists on a finite sequence of maps

$$\begin{array}{ccccccc}
 S_k & \xrightarrow{\sigma_k} & S_{k-1} & \longrightarrow & \dots & \longrightarrow & S_1 & \xrightarrow{\sigma_1} & S_0 \\
 f_k \downarrow & & f_{k-1} \downarrow & & \dots & & f_1 \downarrow & & f_0 \downarrow \\
 V_k & \xrightarrow{\alpha_k} & V_{k-1} & \longrightarrow & \dots & \longrightarrow & V_1 & \xrightarrow{\alpha_1} & V_0
 \end{array}$$

satisfying:

- (i) α_j is the blow-up of V_{j-1} at a singular point p_{j-1} of Z_{j-1} (the branching divisor of f_{j-1}).
- (ii) f_j is the double cover of V_j defined by $\mathcal{L}_j^{\otimes 2} \cong \mathcal{O}(Z_j)$, with $Z_j = \alpha_j^*(Z_{j-1}) - 2m_{j-1}E_j$, $\mathcal{L}_j = \alpha_j^*(\mathcal{L}_{j-1}) \otimes \mathcal{O}_{V_j}(-m_{j-1}E_j)$, where E_j is the exceptional divisor of α_j and p_{j-1} is a singular point of Z_{j-1} of multiplicity $2m_{j-1}$ or $2m_{j-1} + 1$.
- (iii) σ_j is a birational morphism induced by the cartesian diagram of α_j and f_{j-1} .
- (iv) Z_k is smooth and, hence, S_k is a smooth surface.

Now we can use this as follows. Recall from section 1 that we have obtained $f : \tilde{S} \rightarrow V$ a generically 2-to-1 morphism (we can suppose that f is everywhere defined up to blow-ups) from a blow-up of S onto an elliptic fibration V over B which we can suppose relatively minimal after some blow-downs. Suppose that π is relatively minimal.

Now consider

$$\begin{array}{ccccc}
 \tilde{S} & & & & \\
 \sigma \downarrow & \nearrow u & & & \\
 S & \xleftarrow{\quad} & \bar{S} = S_k & \longrightarrow \dots \longrightarrow & S_0 \\
 \pi \downarrow & & f_k \downarrow & & f_0 \downarrow \\
 B & \xleftarrow{\quad} & \bar{V} = V_k & \longrightarrow \dots \longrightarrow & V_0 = V
 \end{array}$$

where:

- $f = f_0 \circ u$ is the Stein factorization of f , with u birational, f_0 finite (so it is a double cover) and S_0 normal.
- $f_k : S_k \rightarrow V_k$ is the canonical resolution of singularities of $f_0 : S_0 \rightarrow V_0$.
- $\bar{\sigma} : S_k \rightarrow S$ is the birational morphism defined by the relative minimality of π .

Theorem 2.1 *Let $\pi : S \rightarrow B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let V be the relative minimal model of the elliptic fibration obtained in section 1. Then*

(a) $\omega_{S/B}^2 - 4\Delta(\pi) \geq 2(g-5)\mathcal{X}\mathcal{O}_V$. In particular, if π is not locally trivial

$$\lambda(\pi) \geq 4 + \frac{2(g-5)\mathcal{X}\mathcal{O}_V}{\Delta(\pi)} \geq 4$$

(b) $\lambda(\pi) = 4$ if and only if S is the minimal desingularization of a double cover $S_0 \rightarrow V$ of a smooth elliptic surface such that

- All the fibres of the elliptic fibration $\tau : V \rightarrow B$ are smooth and isomorphic.
- The branch divisor of the double cover has only negligible singularities (i.e., all the multiplicities m_j in the above process are 2 or 3 (see [13], [17])).

In particular, the bound is sharp.

Proof.

(a) First of all we have

$$\omega_{S/B}^2 - 4\Delta(\pi) = (K_S^2 - 4\mathcal{X}\mathcal{O}_S) - 4(b-1)(g-1) \geq (K_{\bar{S}}^2 - 4\mathcal{X}\mathcal{O}_{\bar{S}}) - 4(b-1)(g-1). \quad (1)$$

For smooth double covers $f_k : \bar{S} \rightarrow \bar{V}$ we have (see [3] p.183):

$$\begin{aligned} \mathcal{X}\mathcal{O}_{\bar{S}} &= 2\mathcal{X}\mathcal{O}_{\bar{V}} + \frac{1}{2}\mathcal{L}_k K_{\bar{V}} + \frac{1}{2}\mathcal{L}_k \mathcal{L}_k \\ K_{\bar{S}}^2 &= 2K_{\bar{V}}^2 + 4\mathcal{L}_k K_{\bar{V}} + 2\mathcal{L}_k \mathcal{L}_k \end{aligned}$$

so we have

$$K_{\bar{S}}^2 - 4\mathcal{X}\mathcal{O}_{\bar{S}} = 2[K_{\bar{V}}^2 - 4\mathcal{X}\mathcal{O}_{\bar{V}}] + 2\mathcal{L}_k K_{\bar{V}}. \quad (2)$$

Moreover, in each blow-up $\alpha_j : V_j \rightarrow V_{j-1}$ we get

$$\mathcal{X}\mathcal{O}_{V_j} = \mathcal{X}\mathcal{O}_{V_{j-1}}; \quad K_{V_j} = \alpha_j^* K_{V_{j-1}} + E_j; \quad \mathcal{L}_j = \alpha_j^* \mathcal{L}_{j-1} - m_{j-1} E_j.$$

Then

$$\begin{aligned} 2[K_{V_j}^2 - 4\mathcal{X}\mathcal{O}_{V_j}] + 2\mathcal{L}_j K_{V_j} &= 2[K_{V_{j-1}}^2 - 4\mathcal{X}\mathcal{O}_{V_{j-1}}] + \\ 2\mathcal{L}_{j-1} K_{V_{j-1}} + 2(m_{j-1} - 1) &\geq 2[K_{V_{j-1}}^2 - 4\mathcal{X}\mathcal{O}_{V_{j-1}}] + 2\mathcal{L}_{j-1} K_{V_{j-1}}. \end{aligned} \quad (3)$$

Finally as $\tau : V \longrightarrow B$ is an elliptic minimal fibration, numerically we have $K_V \equiv \left[2(b-1) + \mathcal{X}\mathcal{O}_V + \sum_i \frac{(n_i-1)}{n_i} \right] E$ ([3] p.162) where E denotes a smooth fibre of τ and $\{n_i\}$ are the multiplicities of singular fibres of τ . In particular $K_V^2 \equiv 0$.

As $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(Z_0)$ and Z_0 is the branch divisor of f_0 we get $\mathcal{L}_0 E = (g-1)$ by Hurwitz formula. So

$$\begin{aligned} 2[K_{V_0}^2 - 4\mathcal{X}\mathcal{O}_{V_0}] + 2\mathcal{L}_0 K_{V_0} &= -8\mathcal{X}\mathcal{O}_{V_0} + \\ + 2\mathcal{L}_0 E \left[2(b-1) + \mathcal{X}\mathcal{O}_{V_0} + \sum_i \frac{(n_i-1)}{n_i} \right] &\geq 4(b-1)(g-1) + 2(g-5)\mathcal{X}\mathcal{O}_V. \end{aligned} \quad (4)$$

Then (a) follows from (1), (2), (3) and (4) and from the fact that $\mathcal{X}\mathcal{O}_V \geq 0$ for elliptic fibrations.

(b) Looking at the proof of (a) we see that $\lambda = 4$ iff $\mathcal{X}\mathcal{O}_V = 0$ and equality holds in (1), (2), (3) and (4). So we have $\lambda = 4$ iff S is the minimal desingularization of a double cover of an elliptic, relatively minimal, fibration $\tau : V \longrightarrow B$ such that:

- τ has no multiple fibres ($\forall i \quad n_i = 1$).
- $\mathcal{X}\mathcal{O}_V = 0$.
- The branch divisor Z_0 of the double cover has only *negligeable singularities* (see [13], [17]), i.e. all the multiplicities of the singularities of the branch divisors in the process of canonical resolution are 2 or 3.

But the first two conditions are equivalent to the fact that τ is smooth and isotrivial (see 15 thms. 6,7 Ch.IV). This allows us to construct examples with $\lambda(\pi) = 4$ which are essentially the same as in [19] example 4.3. So the bound is sharp. \square

Remark 2.2 Although we cannot use double covers for the case of bielliptic fibrations of genus 5 we already know that $\lambda \geq 4$ also holds for such fibrations (see [9] thm.5.1, [11]).

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